Lecture 7:

Decomposition Theorem (cont.), Recurrence

Last Time

Let $\{X_t\}_{t\in\mathbb{N}}$ be a time homogeneous Markov chain, $\mathcal{X}$ be its state space, P be the transition matrix such that $P_{xy} = \mathbb{P}(X_1 = y \mid X_0 = x)$ is the transition probability from the state x to y.

0.1 Define: $\mathbb{P}_x(A) = \mathbb{P}(A \mid X_0 = x)$,

$\mathbb{E}_x(\cdot) = \mathbb{E}(\cdot \mid X_0 = x)$.

the time first visit to x:

$$\tau_x = \min\{n \geq 1 \mid X_n = x\} =: \tau_x^1$$

the time of kth visit to x:

$$\tau_x^k = \min\{n > \tau_x^{k-1} \mid X_n = x\}, \quad \forall k \geq 2.$$

$$\rho_{xy} = \mathbb{P}(\tau_y < \infty \mid X_0 = x) = \mathbb{P}_x(\tau_y < \infty)$$

e.g., $\rho_{yy} = \mathbb{P}_y(\tau_y < \infty)$

0.2 A state x communicates with a state y if $[P^n]_{xy} > 0$ for some $n \geq 1$, which is denoted by $x \longrightarrow y$.

0.3 **Lemma 1**. $x \longrightarrow y$ iff $\rho_{xy} > 0$.

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0.4 Lemma 2. (Transitivity) If $x \to y$ & $y \to z$, then $x \to z$.

0.5 A state x is called transient if $\rho_{xx} < 1$.

A state x is called recurrent if $\rho_{xx} = 1$.

0.6 Thm 1. If $x \to y$ and $\rho_{yx} < 1$, then x is transient.

0.7 Cor 1. If $x \to y$ and x is recurrent, then $\rho_{yx} = 1$.

0.8 We say that T is a stopping time if the occurrence (or nonoccurrence) of the event "we stop at time n," $\{T = n\}$, can be determined by looking at the values of the process up to that time: $X_0, X_1, \cdots, X_n$.

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0.9 $\boxed{\text{Thm 2}}$ (Strong Markov Property).

Suppose T is a stopping time. Given that $T = n$ and $X_T = y$, any other information about $X_0, \cdots, X_T$ is irrelevant for predicting the future. And $\{X_{T+k}\}_{k \in \mathbb{N}}$ behaves like the Markov chain with initial state y.

0.10 A set A is closed if, for any $x \in A$ and $y \notin A$, $P_{xy} = 0$.

A set B is irreducible if, for any $x, y \in B$, x communicates with y (i.e. $x \rightarrow y$, $\forall x, y \in B$).

A set $C \subseteq \mathcal{X}$ is called a communicating class if

1. $\forall x, y \in C$, $x \rightarrow y$ and $y \rightarrow x$.
2. $\forall x \in C$, $y \notin C$, either $x \not\rightarrow y$ or $y \not\rightarrow x$.

0.11. Thm 3. If C is a finite closed and irreducible set, then all states in C are recurrent.

0.12 Thm 4 (Decomposition Theorem).

If the state space $\mathcal{X}$ is finite, then $\mathcal{X}$ can be written as a disjoint union $S \cup R_1 \cup \cdots \cup R_k$, where S is a set of transient states and the R_i, $1 \le i \le k$, are closed irreducible sets of recurrent states.

TODAY

1° Proof of Thm 4.

Let S be the set of x for which there is a y so that $x \longrightarrow y$ but $y \not\rightarrow x$. The states in S are transient by Theorem 1.

To show the rest of the states are recurrent, pick $x \in \mathcal{X} \setminus S$, and denote $C_x := \{y : x \longrightarrow y\}$.

Claim 1: C_x is closed.

i.e. $\forall y \in C_x,\ z \notin C_x,\ P_{yz} = 0$.

Pf. $\forall y \in C_x, \forall z \in \mathcal{X}$ such that $P_{yz} > 0$, we have $y \to z$. Since $x \to y$, Lemma 2 i.e., $\forall y, z \in C_x$, $y \to z$.

$S := \{x \mid \exists y: x \to y, \text{ but } y \not\to x\}$.

implies $x \to z$. That is, $z \in C_x$.

Thus, $z \notin C_x$ implies $P_{yz} = 0$. ■

Claim 2: C_x is irreducible.

Pf. $\forall y, z \in C_x$, since $x \notin S$, $x \to y$ implies $y \to x$. Since $x \to z$, by Lemma 2, we know $y \to z$. Therefore, C_x is irreducible. ■

Now since C_x is a finite closed irreducible set, Thm 3 implies that all the states in C_x are recurrent. Let $R_1 = C_x$. If $(\mathcal{X} \setminus S) \setminus R_1 = \phi$, the proof is complete. Otherwise, pick $x \in (\mathcal{X} \setminus S) \setminus R_1$ and repeat the above procedure. □

2°. In order to prove Theorem 3, we introduce the following two lemmas.

Lemma 3. If x is recurrent and $x \to y$, then y is also recurrent.

Lemma 4. In a finite closed set, there has to be at least one recurrent state.

Recall: Thm 3. If C is a finite closed and irreducible set, then all states in C are recurrent.

Proof of Thm 3. From Lemma 4, there exists $x \in C$, such that x is recurrent. Since C is irreducible, for any $y \in C$, we have $x \to y$. Lemma 3 implies that y is also recurrent. Therefore, all the states in C are recurrent. $\square$

3° Recall. The Strong Markov Property implies

$$\boxed{\mathbb{P}_x(\tau_y^k < \infty) = \rho_{xy} \cdot \rho_{yy}^{k-1}, \quad \forall k \geq 1, \forall x, y \in \mathcal{X}.}$$

Let $N(y)$ be the number of visits to y after the initial time. Then we can compute its expectation.

Lemma 5. $$\mathbb{E}_x N(y) = \begin{cases} 0 & , \rho_{xy} = 0; \\ \dfrac{\rho_{xy}}{1-\rho_{yy}} & , (\rho_{xy} > 0). \end{cases}$$

Proof of Lemma 5. Notice that

$$\mathbb{E}_x N(y) = \sum_{k=1}^{\infty} \mathbb{P}_x(N(y) \geq k)$$

$$= \sum_{k=1}^{\infty} \mathbb{P}_x(\tau_y^k < \infty)$$

$$= \sum_{k=1}^{\infty} \rho_{xy} \cdot \rho_{yy}^{k-1}$$

$$= \frac{\rho_{xy}}{1-\rho_{yy}}. \qquad \square$$

Note. $$\mathbb{E}[X] = \sum_{k=1}^{\infty} k\, \mathbb{P}(X=k) = \sum_{k=1}^{\infty} \mathbb{P}(X \geq k)$$

Note. $\forall\, |x|<1$, $$1 + x + x^2 + \cdots = \frac{1}{1-x}.$$

There is another way of calculating this expectation.

Lemma 6. $\mathbb{E}_x N(y) = \sum_{n=1}^{\infty} [P^n]_{xy}$.

Proof of Lemma 6. Let $\mathbb{1}_{\{X_n=y\}}$ denote the random variable that is 1 when $X_n = y$ and 0 otherwise. Notice that

$$N(y) = \sum_{n=1}^{\infty} \mathbb{1}_{\{X_n=y\}} .$$

Taking expections at both sides gives

$$\mathbb{E}_x N(y) = \mathbb{E}_x\Big[\sum_{n=1}^{\infty} \mathbb{1}_{\{X_n=y\}}\Big] = \sum_{n=1}^{\infty} \mathbb{E}_x[\mathbb{1}_{\{X_n=y\}}]$$
$$= \sum_{n=1}^{\infty} \mathbb{P}_x(X_n = y)$$
$$= \sum_{n=1}^{\infty} [P^n]_{xy} .$$

□

4°. From Lemma 5 and Lemm 6, one can establish the following theorem.

Thm 5 (Equivalent condition of recurrence).

A state $y \in \mathcal{X}$ is recurrent if and only if

$$\sum_{n=1}^{\infty} [P^n]_{yy} = \mathbb{E}_y N(y) = \infty .$$

Proof of Theorem 5. By definition,

y is recurrent

$\iff \rho_{yy} = 1$

Lemma 5 implies $\mathbb{E}_y N(y) = \frac{\rho_{yy}}{1-\rho_{yy}}$

$\iff \mathbb{E}_y N(y) = \infty$

Lemma 6 says $\mathbb{E}_y N(y) = \sum_{n=1}^{\infty} [P^n]_{yy}$

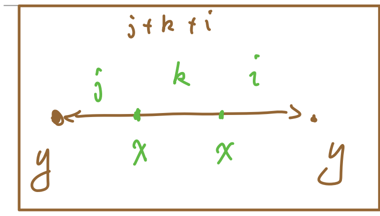
$\iff \sum_{n=1}^{\infty} [P^n]_{yy} = \infty$ $\square$

5°. We now have all the tools to prove Lemma 3 and Lemma 4. Completion of their proofs will fix the gap in the proof of Theorem 3 and thus that in the proof of Theorem 4.

Recall.

Lemma 3. If x is recurrent and $x \longrightarrow y$, then y is also recurrent.

Proof of Lemma 3. Assume x is recurrent and $\rho_{xy} > 0$. By Corollary 1, we know $\rho_{yx} = 1 > 0$. Pick i and j such that $[P^i]_{xy}, [P^j]_{yx} > 0$.



Notice that

$$[P^{j+k+i}]_{yy} \geq [P^j]_{yx} \cdot [P^k]_{xx} [P^i]_{xy}, \quad \forall k \in \mathbb{N}.$$

Therefore, $\sum_{k=1}^{\infty} [P^{j+k+i}]_{yy} \geq \sum_{k=1}^{\infty} [P^j]_{yx} \cdot [P^k]_{xx} [P^i]_{xy}$

$$= [P^j]_{yx} \cdot \left(\sum_{k=1}^{\infty} [P^k]_{xx} \right) [P^i]_{xy}.$$

Since x is recurrent, Theorem 5 implies the right hand side is ∞. So, $\sum_{k=1}^{\infty} [P^{j+k+i}]_{yy} = \infty$ and thus $\sum_{k=1}^{\infty} [P^k]_{yy} = \infty$. Then, by Theorem 5, we know that y is also recurrent. $\square$

Recall

Lemma 4. In a finite closed set, there has to be at least one recurrent state.

Proof of Lemma 4. (Proof by contradiction).

Suppose all the states in a finite closed set C are transient. Then Lemma 5 implies

$$\mathbb{E}_x N(y) < \infty \quad \text{for all } x, y \in C.$$

$$\mathbb{E}_x N_y = \frac{\rho_{xy}}{1-\rho_{yy}}$$

Since C is finite, $\sum_{y\in C} \mathbb{E}_x N(y) < \infty$, $\forall x \in C$.

Applying Lemma 6, one has

$$\infty > \sum_{y\in C} \mathbb{E}_x N(y) = \sum_{y\in C} \sum_{n=1}^{\infty} [P^n]_{xy} = \sum_{n=1}^{\infty} \sum_{y\in C} [P^n]_{xy} \quad (*).$$

Claim: $\forall z \notin C$, x could not reach z from a sequence of arrows "$\longrightarrow$".

Pf. (Proof by contradiction) Suppose not, then $\exists\, x_1, \cdots, x_k \in \mathcal{X}$, s.t. $x \to x_1 \to \cdots \to x_k \to z$.

Let $x_{k+1} := z$ and $K = \max\{i \geq 1 \mid x_i \in C\}$. Then $P_{x_K x_{K+1}} > 0$, where $x_K \in C$ and $x_{K+1} \notin C$. This is a contradiction because C is closed. $\square$

Then, together with Lemma 4 from Lecture 6, this claim implies that $\forall z \notin C$, $x \not\longrightarrow z$. That is, $\forall z \notin C$, $\forall n \geq 1$, $[P^n]_{xz} = 0$. Therefore, $\forall n \geq 1$,

Recall $x \longrightarrow z$ iff $\exists n \geq 1$, s.t $[P^n]_{xz} > 0$.

$$\sum_{y\in C}[P^n]_{xy} = \sum_{y\in C}[P^n]_{xy} + \sum_{z\notin C}[P^n]_{xz} = \sum_{w\in \mathcal{X}}[P^n]_{xw} = 1.$$

Thus, $(*)$ implies

$$\boxed{\infty > \sum_{n=1}^{\infty}\sum_{y\in C}[P^n]_{xy} = \sum_{n=1}^{\infty} 1 = \infty .}$$

This is a contradiction. Thus there has to be at least one recurrent state in C. $\square$

This is the end of this lecture!

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